Lecture notes on global observables

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Contents

1 Introduction

Heavy ion collisions are done in order to produce matter at extreme conditions. Different from high energy physics experiments, we are not after new particles but after properties of matter such as the equation of state, transport coefficients, mean free path, effective degrees of freedom etc. Therefore, methods and tools from solid state physics are equally important as those from high energy physics.

Let us start the discussion by asking ourselves how we characterize a piece of matter (material). First we look at its color, its shape, its weight. Then we squeeze it and scratch it, use a magnet. Next we might drill a hole in it to see what is inside, we might heat it up, put it into an x-ray machine etc. In heavy ion physics we do similar things: We measure spectra (color?), flow (squeeze), excitation functions (heat) jet measurements and correlations (x-ray) and possibly fluctuations (squeeze, magnetic field).

Theoretically, the simplest approach is statistical physics. Here the only assumptions are the conservation laws. But even that can be a daunting task, as the full interaction may not be tractable. Thus, one starts with the ideal gas and slowly introduces more physics. The first correction may be quantum statistics, but in case of HI collisions these are not that essential. More important are interactions and the correlations resulting from them. A simple way to include some of these is to treat resonances as explicit particles and by introducing mean field forces. On a more sophisticated level one could imagine treating correlations explicitely. Also, at lower energies, conservation laws such as strangeness need to be introduced explicitely leading to the canonical treatment.

The quality of the model can be tested against Lattice QCD (LQCD), which gives all thermodynamic quantities in full QCD. A given statistical model should reproduce the equation of state, i.e. entropy and pressure, but also the higher moments of the partition function. Thus, fluctuations are a valuable constraint for the model Hamiltonian. These can also be tested in experiment, at least in principle.

The next more complex level of description introduces dynamics. Again several level of approximation are possible. The two simplest ones are free

streaming corresponding to no interactions and Hydrodynamics, which is based of current conservations under the assumption of local thermal equilibrium. Thus (ideal) hydro is a direct map of the statistical description to the dynamics of the system. All that is needed is the equation of state and some initial conditions. If one wants to compare with spectra, additional assumption concerning the freeze out have to be made as well. The next level of sophistication is viscous hydro and then transport theory. In this case information about equilibration time scales enter the problem. In case of viscous hydro, the transport coefficients which can be obtained in a relaxation time approximation enter. In case of transport the full matrix elements enter and not just the first moments. Thus contrary to hydro, a transport description requires knowledge/assumptions about the relevant degrees of freedom and their interaction. This might be difficult to come by but on the other hand the nature of system may require it.

In these lectures we will essentially walk through all these theoretical descriptions from simple to more sophisticated. At each point we will try to make contact with observables and data and will discuss the limitations of a given theoretical framework.

2 Statistical approach

2.1 Micro-canonical ensemble

Lets start out by reviewing the basic ideas behind statistical mechanics. A macroscopic system is characterized by its conserved quantities, such as energy, momentum, charge, volume etc. These quantities are also referred to as extensive quantities, which scale with the size of the system. Given these conserved quantities, the basic assumption of statistical physics is that *each state with the same value for the conserved quantities is equally probable.* For simplicity, let us assume that the energy is the only conserved quantity. In this case we can write [1]

$$
\begin{aligned}
P_r &= c \ \text{if} E < E_r < E + \delta E \\
P_r &= 0 \ \text{otherwise}
\end{aligned}
\tag{1}
$$

the constant probability c is then given

$$
c = \frac{1}{\Omega(E)}
$$

where $\Omega(E)$ is the number of accessible states for a system with energy E

$$\Omega(E) = \sum_r \delta(E - E_r)\delta E \tag{2}$$

Obviously, this ensures that

$$\sum_r P_r = 1$$

The number of states $\Omega(E)$ is related to the entropy of the system by

$$S(E) = \ln(\Omega(E)) \tag{3}$$

and $\Omega(E)$ is sometimes referred to as the *micro-canonical* partition function. Yes, we are dealing here with the so called *micro-canonical* ensemble, where the system is fully characterized by its extensive (conserved) quantities. Note that $\Omega(E)$ depends on the magnitude of the energy interval δE. As we will show below, for any macroscopic system, thermodynamic quantities like the entropy are independent of the actual size of δE as long as $\delta E \ll E$.

The generalization to more than one conserved quantity is straightforward

$$\Omega(E, Q^i) = \sum_r \delta(E - E_r)\delta E \prod_i \delta_{Q^i, Q^i_r} \tag{4}$$

where the Q^i is are the conserved charges such as electric charge, baryon-number, strangeness etc.

In the micro-canonical ensemble the expectation value of any operator/quantity is simply given by

$$\left\langle \hat{O} \right\rangle = \sum_r P_r \langle r|\hat{O}|r \rangle$$

where $\langle r|\hat{O}|r \rangle$ is the expectation value of the operator $\hat{O}$ for the state $|r>$. Note that we use here the same notation $\langle \rangle$ for quantum and statistical averages. Subsequently we will mostly deal with statistical averages and thus $\langle \rangle$ will refer to statistical averages if not stated otherwise.

By now the obvious questions arise: Where is the temperature, where are the chemical potentials? Isn't that what we use in heavy ion collisions? Why bother with the micro-canonical ensemble? These are valid questions and we will introduce temperature and chemical potentials is due course. But before we get to this, let us discuss the micro-canonical ensemble a little more.

One question which comes to mind is: Where in Eq.4 does the physics enter, in our case, where does QCD enter? Well, the interactions, QCD in our case, affects how many states the system has for a given energy, charge etc. This is it!

Next, let us work out a few simple examples as they will come handy later in the discussion. First, let us calculate the micro-canonical partition function for non-relativistic particles in a box.

2.1.1 Particles in a box

Let us begin by first considering one particle in a box, then turn to many distinguishable particles and finally do the case for indistinguishable particles. The energy eigenvalues for one particle in a box of length L are given by

$$E_n(n_x, n_y, n_z) = \frac{(p_x^2 + p_y^2 + p_z^2)}{2m} = \frac{(2\pi)^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2) \tag{5}$$

with

$$
\begin{aligned}
p_{x,y,z} &= \frac{2\pi}{L} n_{x,y,z} \\
n_x, n_y, n_z &= 1, \ldots \infty
\end{aligned}
$$

the quantum numbers for the motion in the x,y,z direction. With these energies, Eq.5, the micro-canonical partition function Eq.4 can be evaluated. However, its is more instructive and also more realistic to go to the continuum limit, state$L \gg \frac{1}{m}$ where we can convert sums into integrals. In this case

$$\Delta n_x = \frac{L}{2\pi} dp_x$$

and

$$\sum_{n_x,,n_y,n_z} = L^3 \int \frac{dp_x}{2\pi} \frac{dp_y}{2\pi} \frac{dp_z}{2\pi} = V \int \frac{d^3p}{(2\pi)^3}. \tag{6}$$

In this case, Eq. 4 can be written as $(V = L^3)$

$$\Omega(E) = \frac{V}{(2\pi)^3} \int dp_x dp_y dp_z \, \delta(E - E(p))\delta E \tag{7}$$

or

$$\rho(E) \equiv \frac{\delta\Omega(E)}{\delta E} = \frac{V}{(2\pi)^3} \int dp_x dp_y dp_z \, \delta(E - E(p)) \tag{8}$$

where $\rho(E)$ is called *the density of states*, i.e. the number of states in a given energy interval δE. Since

$$E(p) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

the δ-function simply restricts the integral to the surface of a sphere in three dimensions. Taking into account the Jacobian for the δ-function

$$\delta(E - \frac{p^2}{2m}) = \frac{m}{\sqrt{2mE}}\delta(\sqrt{2mE} - |p|)$$

Eq.8 turns into

$$\rho(E) = \frac{V}{(2\pi)^3}\frac{m}{\sqrt{2mE}}\int d^3p\,\delta(\sqrt{2mE} - |p|)$$

where the remaining integral is just the surface of a 2-sphere with radius $R = \sqrt{2mE}$. Thus we get for the density of states

$$\rho(E) = \frac{V}{(2\pi)^3}\frac{m}{\sqrt{2mE}}4\pi(\sqrt{2mE})^2 = \frac{V}{(2\pi^2)}(2m)^{3/2}\sqrt{E} \qquad (9)$$

Now we are ready to consider an arbitrary number N of particles in a box of length L. For each particles we have to sum/integrate over all states subject to the constraint that the total energy is E. Thus we have to deal with an $3N$-dimensional integral for the density of states

$$\rho(E) = \frac{V^N}{(2\pi)^{3N}}\int\prod_{i=1}^{N} dp_i^3\,\delta(E - \frac{1}{2m}\sum_{i=1}^{N}p_i^2)$$

Obviously, the δ-function restricts the area of integration to the surface of a sphere in $3N$-dimensions. The Jacobian for the δ-function is the same as before and we get

$$\rho(E) = \frac{V^N}{(2\pi)^{3N}}\frac{m}{\sqrt{2mE}}\int d^{3N}p\,\delta(\sqrt{2mE} - |p|)$$

The surface of a sphere in n-dimensions of radius R is given by

$$A(S^n) = 2\frac{\pi^{n/2}}{\Gamma(n/2)}R^{n-1}$$

where Γ is Euler's Gamma function. Thus we get for the density of states of N-particles in a box

$$\rho(E) = V^N \Gamma(\frac{3N}{2}) \frac{m^{\frac{3N}{2}}}{(2\pi)^{\frac{3N}{2}}} E^{\frac{3N}{2}-1}$$

Consequently, the density of states grows like large power of the energy $\rho(E) \sim E^{\frac{3N}{2}}$ and *exponentially* with the number of particles. The entropy, which is given by

$$S(E) = \ln \Omega = \ln(\rho \delta E) = (\frac{3N}{2} - 1) \ln(E V^{1/3}) + \ln(\delta E\, V^{1/3}) + \text{constants.}$$
(10)

Coming back to the ambiguity of the choice of δE discussed previously, it is clear from Eq.10 that as long as $\delta E \ll E$ and $N \gg 1$ the term containing δE is a negligible contribution to S and a small change to the size of δE does not change the value of the entropy. As a result, for any macroscopic system ($N \gg 1$) the entropy is controlled by the density of states, i.e.

$$S \sim \ln(\rho V^{-1/3}).$$

2.2 Canonical approximation

2.2.1 Mathematical derivation

Now this was all fine, but we were lucky that we could do the integral of Eq.7 because non-relativistic free particles have such a nice dispersion relation. Already the treatment of relativistic particles gets complicated, because the relation between energy and momentum is not so simple anymore. It would be much more convenient if we found a way to calculate the density of states without having to deal with this δ-function. While δ-functions appear easy to integrate over, this is in fact not the case. If we wanted to do this integral numerically, say by Monte-Carlo methods, we would have to check always that the δ-function is satisfied. It would be much better to have a proper weighting function such that we could sample over all possible states without checking the energy δ-function all the time. Let's see if we can find something like that[1]. Lets start from the original form for the micro-canonical partition function Eq.2

$$\Omega(E) = \sum_r \delta(E - E_r) \delta E$$
(11)

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[1]The are several alternative methods discussed in [1] which all lead to the same result.

and use a suitable representation of the δ-function

$$\delta(E - E_r) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\beta' e^{i\beta'(E-E_r)} e^{\beta(E-E_r)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\beta' e^{\bar{\beta}(E-E_r)} \quad (12)$$

where

$$\bar{\beta} = \beta + i\beta'$$

is complex. Inserting into the expression for $\Omega(E)$,Eq. 11we get

$$\Omega(E) = \frac{\delta E}{2\pi} \sum_r \int_{-\infty}^{+\infty} d\beta' e^{\bar{\beta}(E-E_r)} = \frac{\delta E}{2\pi} \int_{-\infty}^{+\infty} d\beta' e^{\bar{\beta}E} Z(\bar{\beta}) \quad (13)$$

with

$$Z(\bar{\beta}) = \sum_r e^{-\bar{\beta}E_r} = \sum_r e^{-(\beta-i\beta')E_r} \quad (14)$$

Now let us look at this new object, $Z(\bar{\beta})$, more closely. The astute reader will already guess where we are heading. First we note that so far the transformations have been *exact* and that we succeeded in being able to sum over all states without any restriction. The prize we pay is that we have to sum up complex numbers $e^{-(\beta-i\beta')E_r}$. We first note that the imaginary part of $Z(\bar{\beta})$is an odd function of β' and hence does not contribute to $\Omega(E)$. Second, for finite values of β', the expression $e^{-(\beta-i\beta')E_r}$picks up a phase which averages to zero when we sum over many values of E_r. This is illustrated in Fig.1, where we plot the real part of $\sum_{n=1}^{50} e^{i\beta'n}$ as a function of β'. Obviously the only sizable contribution is around $\beta' \simeq 0$.

Let us, therefore, expand the integrant in Eq.13, or rather its logarithm as this is smoother, around $\beta' \sim 0$.

$$
\begin{aligned}
\ln(e^{\bar{\beta}E} Z(\bar{\beta})) &= \bar{\beta}E + \ln(Z(\bar{\beta})) \\
&= (\beta + i\beta')E + \ln(Z(\beta)) + (i\beta')B_1 + \frac{1}{2}(i\beta')^2 B_2 + \ldots \\
&= \beta E + \ln(Z(\beta)) + i(E + B_1)\beta' - \frac{1}{2}B_2\beta'^2 + \ldots
\end{aligned}
$$

with

$$B_k = \left. \frac{\partial^k \ln Z(\bar{\beta})}{\partial \bar{\beta}^k} \right|_{\beta'=0} = \frac{\partial^k \ln Z(\beta)}{\partial \beta^k}.$$

Exponentiating gives then

$$e^{\bar{\beta}E} Z(\bar{\beta}) \simeq e^{\beta E} Z(\beta) e^{-\frac{1}{2}B_2\beta'^2} e^{i(E+B_1)\beta'}.$$
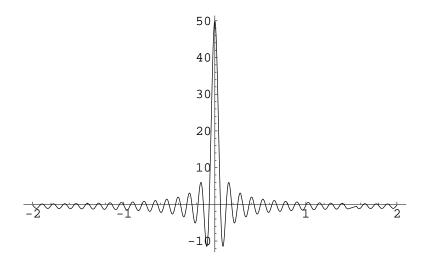
9

Figure 1: Plot of $\Re(\sum_{n=1}^{50} e^{i\beta' n})$as a function of β'.

We note that $|e^{\bar{\beta}E}Z(\bar{\beta})| \sim e^{-\frac{1}{2}B_2\beta'^2}$and thus has a maximum at $\beta' = 0$. Next we can use the parameter β to improve our approximation by choosing β such that the first term in the Taylor expansion above vanishes

$$
\begin{aligned}
E + B_1 &= 0 \\
E + \frac{\partial \ln Z(\beta)}{\partial \beta} &= 0
\end{aligned}
\tag{15}
$$

As a result the phase $e^{i(E+B_1)\beta'}$in the above expression vanishes around $\beta' = 0$ and we have

$$
e^{\bar{\beta}E}Z(\bar{\beta}) \simeq e^{\beta E}Z(\beta)e^{-\frac{1}{2}B_2\beta'^2}
$$

So what we did in practice, was to approximate the oscillatory curve depicted in Fig.1 by a Gaussian, and by choosing the parameter β according to the condition 15 we made sue that the contribution from the oscillatory wings to the integral is minimal. Inserting above expression into Eq.13 we then get for the micro-canonical partition function

$$
\Omega(E) = e^{\beta E}Z(\beta)\frac{\delta E}{2\pi} \int_{-\infty}^{+\infty} d\beta' e^{-\frac{1}{2}B_2\beta'^2} = e^{\beta E}Z(\beta)\frac{\delta E}{\sqrt{2\pi B_2}}
$$

and consequently

$$
\ln \Omega(E) \simeq \beta E + \ln Z(\beta).
\tag{16}
$$

Does this look familiar? If not, recall that the entropy was defined as $S = \ln \Omega$. Further, let's just for the "fun" of it, define

$$
\begin{aligned}
\beta &\equiv \frac{1}{T} \\
\ln Z &\equiv -\frac{F}{T}.
\end{aligned}
\tag{17}
$$

With these definitions Eq.17 can be written as

$$
F = E - TS.
\tag{18}
$$

Now this looks very familiar. Just like the definition of the Helmholtz free energy. Somebody might argue that this cannot be true. We just did some mathematics and approximation. How come that we all of a sudden wind up with physical quantities such as temperature and free energy? Well, lets first look at the free energy or rather at the function Z. From its definition (Eq.14) we find

$$
Z(\beta) = \sum_r e^{-\beta E_r}.
\tag{19}
$$

So, if β is indeed related to the temperature as given by Eq.15, then this indeed looks like the well know *canonical* partition from statistical physics. Lets us further look at the condition for β in order to maximize the quality of our approximation (Eq.15). Using the definitions of 17 we find

$$
0 = E + \frac{\partial \ln Z}{\partial \beta} = E - \frac{\partial (F/T)}{\partial (1/T)} = E + T \frac{\partial F}{\partial T} - F
$$

or

$$
E = F - T \frac{\partial F}{\partial T}
$$

which is again a well know formula. If we further insert the basic thermodynamic relation 18 we also find that

$$
S = -\frac{\partial F}{\partial T}.
$$

Finally, if we insert Eq.19 into Eq.15 we get

$$
E = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}}
$$

which is the canonical expression for the mean energy, suggesting the the probability to find a state r is given by

$$P_r = \frac{e^{-\beta E_r}}{\sum_r e^{-\beta E_r}}.$$

Below (Eq.21) we will show that this is indeed the case. Obviously, our mathematics just gave us a "thermodynamic" system in the canonical description, which usually requires the coupling to a heat-bath and allowing for exchange of energy between the system and heat-bath.

2.2.2   Physical derivation

Let us briefly review how the standard argument goes when introducing the canonical description. First one considers a system $\mathcal{S}$ and a reservoir $\mathcal{R}$ (heat-bath) such that the reservoir is much bigger than the system, i.e

$$\Omega_\mathcal{S} \ll \Omega_\mathcal{R}$$

next one allows for exchange of energy between the two systems and maximizes the entropy of the combined system. The number of accessible states of the combined system is

$$\Omega_{total}(E) = \sum_{\Delta E} \Omega_\mathcal{S}(\Delta E)\Omega_\mathcal{R}(E - \Delta E)$$

As we have shown above $\Omega(E)$ is a steeply rising function of E, and thus $\Omega(E_0 - E)$ is equally steeply falling. Consequently, the product $\Omega_\mathcal{S}(\Delta E)\Omega_\mathcal{R}(E - \Delta E)$ has a sharp maximum as a function for ΔE. Thus the sum will be dominated by a few terms, and the most probable configuration is given by the maximum of $\Omega_\mathcal{S}(\Delta E)\Omega_\mathcal{R}(E - \Delta E)$. Finding the maximum by setting the derivative with respect to ΔE to zero we obtain

$$
\begin{aligned}
0 &= \frac{\partial}{\partial(\Delta E)}[\Omega_\mathcal{S}(\Delta E)\Omega_\mathcal{R}(E - \Delta E)] \\
0 &= \frac{\partial \Omega_\mathcal{S}}{\partial E}\Omega_\mathcal{R} - \frac{\partial \Omega_\mathcal{R}}{\partial E}\Omega_S \\
\frac{1}{\Omega_\mathcal{S}}\frac{\partial \Omega_\mathcal{S}}{\partial E} &= \frac{1}{\Omega_\mathcal{R}}\frac{\partial \Omega_\mathcal{R}}{\partial E} \\
\frac{\partial}{\partial E}\ln \Omega_\mathcal{S} &= \frac{\partial}{\partial E}\ln \Omega_\mathcal{R}
\end{aligned}
$$

using the definition for the entropy $S = \ln \Omega$ we find that the derivatives of the entropy with respect to the energy in both systems should be the same

$$\frac{\partial}{\partial E} S_{\mathcal{S}} = \frac{\partial}{\partial E} S_{\mathcal{R}}.$$

Since two systems in thermal contact are in equilibrium if their temperatures are the same, this suggests the definition for the temperature as

$$\frac{1}{T} = \beta = \frac{\partial}{\partial E} S(E). \tag{20}$$

Defining the inverse of the temperature to be equal to the energy derivative of the entropy ensures that the energy flows from a system of higher temperature to that of lower temperature [1]. This statistical definition of the temperature is very powerful, and indeed if we apply the definition 20 to what we found by doing pure math, Eq.16, we see that indeed our intuition that $\beta = \frac{1}{T}$ was correct.

Let us now introduce the Boltzmann factors following the same line of reasoning. To this end we ask ourselves, what is the probability that the system $\mathcal{S}$ is in a state n with energy E_n. Following Eq.1 this probability is given by

$$P_n = \frac{\Omega_{\mathcal{R}}(E_{total} - E_n)}{\Omega_{tot}(E_{total})}$$

since for a fixed state r of the system we have $\Omega_{\mathcal{R}}(E_{total} - E_n)$ states of the reservoir each with the same probability $1/\Omega_{tot}(E_{total})$. Therefore, the ratio of probabilities for the system to be in states n and m is

$$\frac{P_n}{P_m} = \frac{\Omega_{\mathcal{R}}(E_{total} - E_n)}{\Omega_{\mathcal{R}}(E_{total} - E_m)}.$$

Taking the logarithm and expanding for $E_{n,m} \ll E_{total}$ we get

$$\ln(\frac{P_n}{P_m}) = S_{\mathcal{R}}(E_{total} - E_n) - S_{\mathcal{R}}(E_{total} - E_m) \simeq \frac{\partial S_{\mathcal{R}}}{\partial E}|_{E=E_{total}}(E_m - E_n)$$

or

$$\frac{P_n}{P_m} \simeq \frac{e^{-\beta E_n}}{e^{-\beta E_m}} = \frac{e^{-E_n/T}}{e^{-E_m/T}}.$$

Consequently the probability to find a state n is given by

$$P_n = \frac{e^{-\beta E_n}}{\sum_m e^{-\beta E_m}} = \frac{1}{Z} e^{-\beta E_n} \tag{21}$$

which incidentally is exactly what we concluded from our mathematical exercise above.

2.2.3 Some "academic" example: Exponential density of states

Let us look at the rather "unusual" case of an exponential density of states

$$\rho(E) = e^{\frac{E}{\alpha}}$$

In this case the temperature is given by (see Eq.20)

$$T = \left(\frac{\partial \ln(\rho)}{\partial E}\right)^{-1} = \left(\frac{\partial}{\partial E}\frac{E}{\alpha}\right)^{-1} = \alpha = constant$$

INDEPENDENT of the energy. To put this in perspective, for an ideal gas we have

$$T = \frac{2}{3}\frac{E}{N}$$

i.e. the temperature is proportional to the energy. In case of the ideal gas, we kow that the density of states grows like a power law

$$\rho(E) \sim E^{\frac{3N}{2}}$$

i.e. slower than exponential. Imagine that the density of states would grow faster than exponential such as for example

$$\rho(E) \sim e^{\gamma(E)E}$$

with $\gamma(E)$ a function increasing with E, for instance $\gamma(E) \sim E^n$. In this case the temperature would be

$$T(E) \sim \frac{1}{\gamma(E)} \sim E^{-n}.$$

Consequently the temperature would *decrease* with increasing energy, leading to inconsistent thermodynamic bhaviour. Thus we conclude, that the density of states can grow at most exponentially with the energy.

Why did we bother with this academic case? Well, there is the so-called Hagedorn model, which assumes that the density of hadronic states grows exponentially with the mass

$$\rho(m) \sim e^{\frac{m}{T_H}} \tag{22}$$

where $T_H \sim 170\,\mathrm{MeV}$ is the Hagedorn temperature. For hadron-masses $m \geq 1\,\mathrm{GeV}$ density of states given by Eq.22 actually is well supported by the

experimentally determined spectrum of hadronic states. Let us assume for a moment that Eq.22 is valid for all masses. Let us next consider on particle in a box which is subject to the Hagedorn mass spectrum. Staying with a non-relativistic description the density of states of this system is given by

$$\rho_H(E) = \int dm \int_0^E dE_k \, e^{\frac{m}{T_H}} \rho_0(E_k) \, \delta(E - (m + E_k)).$$

Here, $\rho_0(E)$ stands for the density of states of one particle in a box, Eq.9. Integrating over the mass and using the δ-function, we get for the density of states

$$\rho_H(E) = e^{\frac{E}{T_H}} \int_0^E dE_k \, e^{\frac{mE_k}{T_H}} \rho_0(E_k)$$

and

$$S_H(E) = \ln(\rho_H(E)) = \frac{E}{T_H} + \ln(I(E)), \tag{23}$$

where $I(E)$is simply the integral in the previous equation

$$I(E) = \int_0^E dE_k \, e^{\frac{mE_k}{T_H}} \rho_0(E_k).$$

Now for $E \gg T_H$ the first term in Eq.23, which is linera in E, will be much bigger than the second term which grows only logarithmically. In this case

$$S_H(E) \sim \frac{E}{T_H}.$$

As a result the temperature of such a system for ALL energies $E \gg T_H$is constant, independent of the energy and given by the Hagedorn temperature, T_H

$$T(E) = T_H, \, , \, E \gg T_H.$$

Indeed, we predict a limiting temperature and this is seen in the analysis of hadronic ratios.

2.3 Grand canonical ensemble, single particle partition functions etc.

2.3.1 General considerations

Before we discuss the implications for the heavy ion physics, let us introduce a few more well known concepts from statistical physics. We shall be brief

here, as most of the ideas are analogous to what we have discussed in the previous sections and details can be found in standard textbooks such as [1].

Let us start out with the canonical partition for a single particle in a box

$$Z_1 = V \int d^3p \, e^{-\beta E(p)} \tag{24}$$

The partition function for N *non-interacting distinguishable* particles is simply the product of the single particle partition functions

$$Z_{N,distin.} = Z_1^N$$

In case of *indistinguishable* particles, we have to divide by all permutations which leave the configuration unchanged

$$Z_N = \frac{Z_1^N}{N!} \tag{25}$$

Subsequently Z_N will always refer to the *indistinguishable N-particle* partition function. These formulas will come in handy when we will discuss grand-canonical ensemble in more detail and when effects of explicit charge/strangeness conservation will be addressed. Let us now turn to the grand-canonical ensemble.

As already discussed, a macroscopic system is characterized not only by its energy but also by conserved charges, such as the electric charge, strangeness, baryon-number etc. Just as in the transition from the micro-canonical to the canonical description, where we gave up on the exact conservation of energy, one introduces the so-called *grand-canonical* ensemble, where one keeps the conserved charges fixed only on the average. For systems with large number of charges this should be a good approximation[2] and again, this can be viewed either as mathematical approximation or as a system in contact with a reservoir, where now also the exchange of conserved charges is allowed. The corresponding "Boltzmann"-actor then becomes

$$e^{-\beta(E_r - \sum_i \mu_i Q_r^i)}$$

and the grand-canonical partition function given the conserved charges Q^i and associated chemical potentials μ_i is given by

$$Z_G = \sum_{all\ states\ r} e^{-\beta(E_r - \sum_i \mu_i Q_r^i)} \tag{26}$$

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[2]This is true for averages only. For variances extra care needs to be taken as discssed below.

Here the sum includes states with different particle number. The probability to find a state r is given by

$$P_r = \frac{1}{Z_G} e^{-\beta(E_r - \sum_i \mu_i Q_r^i)}. \tag{27}$$

The corresponding free energy, which is often called the *"thermodynamic potential"* is again given by the log of the partition function

$$F_G = -T \ln Z_G \tag{28}$$

Lattice QCD at finite temperature for instance does nothing else but calculating the grand-canonical partition function for QCD, at present only for vanishing chemical potentials. Given this partition function, observables such as the average charge Q_i is obtained by differentiating with respect to the appropriate conjugate variable, in this case the chemical potential μ_i

$$\langle Q_i \rangle = \frac{\sum_{all\,states\,r} Q_i \, e^{-\beta(E_r - \sum_i \mu_i Q_r^i)}}{\sum_{all\,states\,r} e^{-\beta(E_r - \sum_i \mu_i Q_r^i)}} = \frac{1}{\beta} \frac{\partial}{\partial \mu_i} \ln Z_G. \tag{29}$$

Variances and co-variance are second derivatives of $\ln Z_G$

$$\left\langle (\delta Q_i)^2 \right\rangle \equiv \left\langle Q_i^2 \right\rangle - \left\langle Q_i \right\rangle^2 = \frac{1}{\beta^2} \frac{\partial^2}{\partial \mu_i^2} \ln Z_G \tag{30}$$

$$\langle \delta Q_i \delta Q_j \rangle = \langle Q_i Q_j \rangle - \langle Q_i \rangle \langle Q_j \rangle = \frac{1}{\beta^2} \frac{\partial^2}{\partial \mu_i \partial \mu_j} \ln Z_G \tag{31}$$

and we will used these expressions when we come to the discussion of fluctuations.

2.3.2 Example: ideal gas

To illustrate these concepts and to introduce a few new, often used formulae, let us discuss the grand canonical description of an ideal gas. We will start out with the conventional approach and then use a more "modern" approach, which considers the ideal gas as a sum of sub-systems, where each subsystem is a single particles level (orbital). Suppose, that we consider a system where the number of particles N is a conserved quantity. Then the Z_G (Eq.26) reads

$$Z_G = \sum_{all\,states\,r} e^{-\beta(E_r - \mu N)}.$$

17

In case of a classical ideal gas, where we have no interactions, the sum over all states can be decomposed into a sum over system with a given particle number N times the sum over all energies given this particle number

$$Z_G = \sum_N \sum_{\epsilon_N} e^{-\beta(\epsilon_N - \mu N)} = \sum_N e^{\beta \mu N} \sum_{\epsilon_N} e^{-\beta \epsilon_N}.$$

The second sum is nothing but the N-particle canonical partition function Z_N, Eq.25 so that

$$Z_G = \sum_N e^{\beta \mu N} Z_N = \sum_N \frac{(e^{\beta \mu} Z_1)^N}{N!} = \exp(e^{\beta \mu} Z_1)$$

where we have used the relation between Z_N and Z_1 for indistinguishable particles. As we have seen above, the quantity of relevance is the logarithm of Z_G or the free energy

$$\ln Z_G = e^{\beta \mu} Z_1 = V \int \frac{d^3 p}{(2\pi)^3} \, e^{-\beta(E(p) - \mu)} \equiv Z_{1,G}.$$

Here we have defined the grand-canonical single-particle partition function $Z_{1.G}$ in analogy to Z_1, Eq.24. This already looks familiar. Lets us next calculate the mean number of particles. From Eq.30 we get

$$\langle N \rangle = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Z_G = V \int \frac{d^3 p}{(2\pi)^3} \, e^{-\beta(E(p) - \mu)} = Z_{1,G}. \tag{32}$$

Now this is something we all know. In order to obtain the number of particles, just integrate over the "Boltzmann"-factor over all momenta. To get a better understanding of this result let us look at the "modern" derivation of the ideal gas.

Another, equivalent, way to look at the ideal gas is to consider it as a ensemble of system, where each system is just one individual single particle quantum state (orbital). This has the added advantage that quantum statistics can be easily taken into account as it only affects the occupation of a given state. In this approach, the total partition function is simply the products of the partition functions of all single particle states

$$Z_G = \prod_s Z_s,$$

where s denotes the individual state. So all we need to do is to work out Z_s for a given state. Lets do Bose-Einstein statistics first, then Fermi-Dirac and

finally take the classical limit. For Bose statistics, there is no limit in how many particles we can put into a given state s. If n particles are in the state s its energy is given by $E = n\epsilon_S$, where ϵ_s is the energy level of the single particle state. Thus we have

$$Z_s^{B.E.} = \sum_{n=0}^{\infty} e^{-\beta(n\epsilon_s - n\mu)} = \sum_{n=0}^{\infty} (e^{-\beta(\epsilon_s - \mu)})^n = \frac{1}{1 - e^{-\beta(\epsilon_s - \mu)}}.$$

Given Z_s we can calculate the mean number of particles in the state s

$$\begin{aligned}
\langle n \rangle_s^{B.E.} &= \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Z_s^{B.E.} = \frac{1}{Z_s} \frac{1}{\beta} \frac{\partial}{\partial \mu} Z_s^{B.E.} = \frac{e^{-\beta(\epsilon_s - \mu)}}{1 - e^{-\beta(\epsilon_s - \mu)}} \\
&= \frac{1}{e^{+\beta(\epsilon_s - \mu)} - 1} \equiv f_{B.E.}(\epsilon_s, \mu)
\end{aligned}$$

and find the well know occupation function for Bose-Einstein statistics. Since we are mostly interested in the logarithm of Z_G, we have

$$\ln Z_G = \sum_s \ln Z_s^{B.E.} = V \int \frac{d^3 p}{(2\pi)^3} \ln Z_p = -V \int \frac{d^3 p}{(2\pi)^3} \ln(1 - e^{-\beta(\epsilon_p - \mu)}).$$

Here we used the continuum limit, Eq.6, and we label the states with by their momenta. As a consequence of this well know expression, the total particle number $\langle N \rangle$ is given by

$$\langle N \rangle^{B.E.} = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Z_G^{B.E.} = V \int \frac{d^3 p}{(2\pi)^3} \langle n \rangle_p = V \int \frac{d^3 p}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1},$$

i.e. simply by the sum over the mean occupation of all states. Similar expression can be derived for the variances as we will discuss later when we talk about fluctuations.

Fermi-Dirac statistics is done in a similar fashion. Given a state s the Pauli-principle allows for no more the one particle in the state, hence

$$Z_s^{F.D.} = 1 + e^{-\beta(\epsilon_s - \mu)}.$$

Going through the same steps as before we find

$$\begin{aligned}
\langle n \rangle_s^{F.D.} &= \frac{1}{e^{\beta(\epsilon_s - \mu)} + 1} \equiv f_{F.D.}(\epsilon_s, \mu), \\
\ln Z_G^{F.D.} &= V \int \frac{d^3 p}{(2\pi)^3} \ln(1 + e^{-\beta(\epsilon_p - \mu)}), \\
\langle N \rangle^{F.D.} &= V \int \frac{d^3 p}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon_p - \mu)} + 1}.
\end{aligned}$$

Both results can be cast into compact form

$$\ln Z_G = \pm V \int \frac{d^3 p}{(2\pi)^3} \ln(1 \pm e^{-\beta(\epsilon_p - \mu)})$$

$$\langle N \rangle = V \int \frac{d^3 p}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon_p - \mu)} \pm 1}$$

where the upper sign refers to Fermi-Dirac and the lower sign to Bose-Einstein statistics, respectively. The classical limit is obtained if $e^{-\beta(\epsilon_p - \mu)} \ll 1$, i.e. the occupation of each orbital is small

$$\ln Z_G^{classical} = V \int \frac{d^3 p}{(2\pi)^3} e^{-\beta(\epsilon_p - \mu)}$$

$$\langle N \rangle^{classical} = V \int \frac{d^3 p}{(2\pi)^3} e^{-\beta(\epsilon_p - \mu)}$$

which is of course our previous result.

2.3.3 Explicit conservation of charges

In the context of heavy ion collisions it has been realized that the conservation of some charges have to be taken into account explicitely, notably strangeness at lower energies (AGS) or possible in peripheral collisions. From what we already know, we expect that this explicit conservation will become relevant only if few quanta carrying the conserved charge are present. In this case, the above mathematical tricks are not well justified as the entropy is not as sharply peaked as it should in order to justify the Gaussian approximation. This is often referred to as the "canonical" limit. Of course this is no limit, as the system should always be treated micro-canonically. And everything else are limits. But what is meant is that in the "limit" of few strange particles, the grand-canonical approximation fails and one has to treat strangeness explicitely while still approximating the energy-conserving δ-function as discussed above. This is a reasonable approach as there is enough energy to go around. Also the electric charge can be treated grand canonically, because there are many charge carriers.

In order to discuss the basic principles, let us consider a system which consists of kaons and anti-kaons only, and let us derive the basic relations assuming that strangeness is conserved explicitely, i.e.

$$S = N_+ - N_-; \quad \delta S = 0$$

Here N_+ denotes the number of kaons, and N_- the number of anti-kaons[3]. Let us again assume that we are dealing with a (classical) ideal gas, i.e no interactions. Then the partition function for N_+ kaons is given by

$$Z_{N_+} = \frac{Z_{1,+}^{N_+}}{N_+!} = \frac{\xi_+^{N_+}}{N_+!}$$

where we introduced a short hand for the one particle partition function $Z_{1,+} \equiv \xi_+$. Note also, as a result of Eq.32, ξ_+ is simply the mean number of kaons in a grand-canonical system with vanishing strangeness chemical potential

$$\xi_+ = \langle N_+ \rangle_{G, \mu_s = 0} \, .$$

Obviously, every time we have N_+ kaons, we need to balance with N_- anti-kaons. Consequently the partition function is

$$Z_{can}(S=0) = \sum_m \frac{\xi_+^n \xi_-^n}{n!n!}$$

Each term in the sum explicitly has strangeness $S = 0$ and we sum over the remaining free variable, namely the number of kaons/anti-kaons. Obviously, this can be generalized to any value of the net strangeness

$$Z_{can}(S) = \sum_{m=S}^{\infty} \frac{\xi_+^{S+m} \xi_-^m}{(S+m)!m!}. \tag{33}$$

It turns out that the above sum is identical to the Bessel function $I_S(z)$ which has the series representation

$$I_S(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(S+k+1)} \left(\frac{z}{2}\right)^{(S+2k)} = \sum_{k=0}^{\infty} \frac{1}{k!(S+k)!} \left(\frac{z}{2}\right)^{(S+2k)}.$$

With

$$z \equiv 2\sqrt{\xi_+ \xi_-}$$

the partition function is then

$$Z_{can}(S) = I_S(z) \left(\frac{\xi_+}{\xi_-}\right)^{S/2}. \tag{34}$$

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[3]Note that for historical reasons the strangeness of a kaon is positive, although it contains and strange *anti*-quark. Thus a strange quark carries strangeness $s_q = -1$.

The average number of kaons is obtained in the usual way. In each term in the sum of Eq.33 the number of kaons is given by the exponent of ξ_+. Therefore,

$$\langle N_+ \rangle = \frac{\sum_{m=S}^{\infty}(S+m)\frac{\xi_+^{S+m}\xi_-^m}{(S+m)!m!}}{\sum_{m=S}^{\infty}\frac{\xi_+^{S+m}\xi_-^m}{(S+m)!m!}} = \frac{1}{Z_{can}(S)}\xi_+\frac{\partial}{\partial\xi_+}Z_{can}(S) = \xi_+\frac{\partial}{\partial\xi_+}\ln Z_{can}(S).$$

Likewise the variance is given by

$$\left\langle (\delta N)^2 \right\rangle = \left(\xi_+\frac{\partial}{\partial\xi_+} \right)^2 \ln Z_{can}(S)$$

In terms of the Bessel function, this translates into

$$\langle N_+ \rangle = \frac{z}{2}\frac{I_{S+1}(z)}{I_S(z)} + S$$

$$\langle N_- \rangle = \frac{z}{2}\frac{I_{S+1}(z)}{I_S(z)}$$

$$z = 2\sqrt{\xi_+\xi_-}. \tag{35}$$

For the case e of interest in heavy ion collisions, $S = 0$, we have

$$\langle N_+ \rangle = \langle N_- \rangle = \frac{z}{2}\frac{I_1(z)}{I_0(z)}.$$

Now for small temperatures and/or small system sizes, $\xi_\pm \ll 1$, hence $z \ll 1$ and we may expand the above expression for small arguments. With

$$I_0(z) = 1 + \frac{z^2}{4} + \mathcal{O}(z^4)$$

$$I_1(z) = \frac{z}{2} + \frac{z^3}{16} + \mathcal{O}(z^4)$$

we get to leading order

$$\langle N_+ \rangle = \langle N_- \rangle \simeq \frac{z^2}{4} = \frac{1}{2}\xi_+\xi_- = \frac{1}{2}V^2 \left(\int \frac{d^3p}{(2\pi)^3} e^{-\beta E} \right)^2. \tag{36}$$

So the number of kaons scales with the volume *squared!* and with the square of the kaon number in the grand-canonical limit. Since we are in the regime where $\xi_+ \ll 1$ we further have

$$\langle N_+ \rangle_{canonical} = \frac{1}{2}\xi_+\xi_- = \frac{1}{2}\xi_+^2 \ll \xi_+ = \langle N_+ \rangle_{grand-canonical}.$$
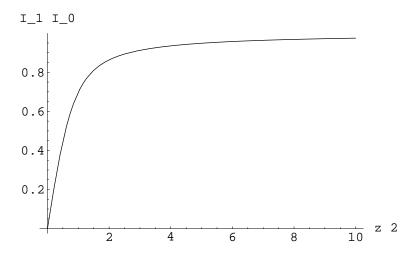
22

```
I_1 I_0
```


0.8

0.6

0.4

0.2

2   4   6   8   10   z   2

Figure 2: Ratio of Bessel-functions I_1/I_0.

As expected the kaon number is suppressed due to explicit strangeness con-
servation. For large values of $\xi_{\pm}$, (see Fig.2)

$$\frac{I_1(z)}{I_0(z)} \to 1; \ z \to \infty$$

so that

$$\langle N_{\pm} \rangle == \frac{z}{2} = \sqrt{\xi_{+}\xi_{-}} = \xi_{\pm} = \langle N_{\pm} \rangle_{grand-canonical}$$

and we recover the grand-canonical value when the number of quanta is
large[4].

From Eq.35 the suppression of the kaon number due to explicit strangeness
conservation is given by the ration of the Bessel functions $\frac{I_1(z)}{I_0(z)}$, while the
grand-canonical kaon number is given by $\xi_{+} = z/2$. Thus the plot in Fig.2
shows the suppression factor as a function of grand-canonical kaon number.
If the number of kaons is larger than $N_k \geq 2$ the grand-canonical approxi-
mation is fine. In the context of heavy ion collisions one might argue that
this is not a problem as one has more than 2 kaons at least for AGS energies
and above. This, however, is too simplistic as we shall see below. The kaon
number of relevance is that within a volume over which the system is fully

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[4]The fact that the grand-canonical and canonical ensemble are equivalent is only true
for mean values. Once we look at variances (fluctuations) this is not longer the case as we
will discuss below.

23

equilibrated. This could be much smaller than the full volume of the fireball, and consequently canonical suppression might be of great importance. This argument can be turned around, and one can use the canonical suppression in order to estimate the size over which the system has truly equilibrated.

After these rather formal developments, let us no turn to the physics of heavy ion collisions and see how the insights gained here allow us to better understand and interpret the data.

2.4 Relation to heavy ion collisions

So why did we do all this? Well we saw that the phase space grows rapidly with energy and particle number and that we can introduce concepts like "temperature" and free energy without ever talking about a heat-bath etc. We also provided the foundation for all the formulae used in statistical hadronization models. We further saw that explicit strangeness conservation is important for small and/or cold systems.

All we did was to approximate the calculations for the number of accessible states by an integral over a Gaussian, which captures most of the integral. In physical terms, we described the ensemble by requiring that the mean energy of this system is the same as the micro-canonical energy via 15.

In other words, if we can describe a statistical ensemble by a temperature, it does not necessarily mean that we a dealing with a "thermal" system in the Boltzmann sense, where particles collide and equilibrate the system. It could simply be generated by "randomly" throwing particles into phase space or by rolling dice fro that matter. This finding will be important when we discuss particle ratios and chemical equilibrium in the context of elementary particle collisions as well as heavy ion collisions.

3 Fluctuation and Correlations

Now that we have gained some understanding of the static properties of the statistical system, lets now turn to fluctutuations. To start lets recall what we learned in the previous chapter. Given a partition function

$$Z = \sum_s \left\langle s | e^{-\beta(\hat{H} - \sum_i \mu_i \hat{Q}_i)} | s \right\rangle = \sum_s e^{-\beta(E_s - \sum_i \mu_i Q_{i,s})} \left\langle s | s \right\rangle,$$

where we have assumed that s denotes Eigenstates of the system with a given charges Q_i[5], we can evaluate the mean of any operator $\hat{O}$

$$\left\langle \hat{O} \right\rangle = \frac{1}{Z} \sum_s e^{-\beta(E_s - \sum_i \mu_i Q_{i,s})} \left\langle s | \hat{O} | s \right\rangle \equiv \frac{1}{Z} tr(\hat{\rho}\hat{O}).$$

Here $\hat{\rho}$ is the statistical operator

$$\hat{\rho} = \sum_s e^{-\beta(\hat{H} - \sum_i \mu \hat{Q}_i)} | s \rangle \langle s | \tag{37}$$

Now we are ready to write down the general expressions for variances and co-variances

$$\left\langle (\delta x)^2 \right\rangle \equiv \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 = \frac{1}{Z} tr(\hat{\rho}\, x^2) - \left[\frac{1}{Z} tr(\hat{\rho}\, x) \right]^2 \tag{38}$$

$$\left\langle \delta x\, \delta y \right\rangle \equiv \left\langle xy \right\rangle - \left\langle x \right\rangle \left\langle y \right\rangle = \frac{1}{Z} tr(\hat{\rho}\, xy) - \frac{1}{Z^2} tr(\hat{\rho}\, x) tr(\hat{\rho}\, y). \tag{39}$$

While these averages can be taken for any quantity, if we want to stick with the philosophy of statistical mechanics, we should stick with the variables available, such as energy, and the charges.

3.1 Energy Fluctuations

3.2 Charge Fluctuations

3.3 Correlations

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[5]This is always possibke since the charges are conserevd and thus commute with the Hamiltonian.

4 Dynamical models

So far our discussion has concentrated on static systems. Obviously this simplifies the discussion and theoretical treatment, but a heavy ion collision is a highly dynamic process. Also, contrary to an elementary particle collision such as e^+e^- it is virtually impossible to calculate the matrix element which connects the initial and final state of such a collision. Already the initial state, two nuclei, is a highly complex many body system and while there might be some hope that one can treat at least the low Bjorken x part of the initial wave function the subsequent evolution is hopeless to address the entire collision as a full quantum process. Therefore one has to resort to approximations which, so far are all classical and semi classical methods.

In this chapter we will discuss two dynamical models, hydrodynamics on the one hand and Boltzmann-transport on the other. We will also show how hydrodynamics results from Boltzmann-transport in the limit of long wavelength and rapid local equilibration.

4.1 Hydrodynamics

Hydrodynamics is similar to the statistical approach in philosophy. It solely relies on conservation laws and the dynamics are simply governed by the conserved currents of the system. In statistical physics the system was characterized by energy and momentum conservation and by all conserved charges. Similarly, in hydrodynamics, the time evolution of the system is controlled by the continuity equations associated with the same conserved quantities, energy/momentum, and conserved charges. For this to work, one has to assume that the system is locally in equilibrium, so that by knowing the value of these conserved quantities at a given point in space and time, we know the properties of the matter at this time. One can relax this requirement, which then leads to viscous hydrodynamics and eventually to transport theory, as we will discuss in the second part of this chapter. If everything is controlled by the conservation laws, including the conservation of entropy, then one speaks of *ideal* hydrodynamics. In this section we will exclusively deal with ideal hydrodynamics and thus will drop the qualifier "ideal" in the subsequent discussion. Since hydrodynamics is based on continuity equations let us remind ourselves what a continuity equation is all about.

4.1.1 Continuity equations

Each conserved charge Q with

$$\frac{d}{dt}Q = 0$$

is associated with a current $\vec{j}(\vec{r})$ such that

$$\frac{\partial}{\partial t}\rho(\vec{r}) + \vec{\nabla}\vec{j}(\vec{r}) = 0 \tag{40}$$

where ρ is the charge density such that

$$\int d^3r\,\rho(\vec{r}) = Q.$$

For example consider dN particles with charge q in a small volume dV. Then the charge density is given by

$$\rho = q\frac{dN}{dV}$$

The total charge in the volume $Q_V = \rho\,dV$ changes as particles move in and out through surface of the volume. If the particles move with a velocity $\vec{v}$ this change during a time interval dt is given by

$$\delta Q_V = dt \oint_{S(dV)} \rho\vec{v}\,d\vec{n}$$

where the integral is over the surface of dV and $\vec{n}$ is a vector normal to the surface pointing outside; $d\vec{n}$ is the corresponding surface differential. On the other hand, the change of the charge is due to the change of the charge density inside the volume

$$\delta Q_V = -dV\frac{\partial \rho}{\partial t}dt.$$

Hence, we have, using Gauss' law,

$$-dV\frac{\partial \rho}{\partial t} = \oint_{S(dV)} \rho\vec{v}\,d\vec{n} = \int dV\vec{\nabla}(\rho\vec{v}) = dV\vec{\nabla}(\rho\vec{v})$$

With the current

$$\vec{j} = \rho\vec{v}$$

27

we arrive at the continuity equation Eq.40. Defining a four-vector

$$j^\mu = (\rho, \vec{v}\rho)$$

the continuity equation can be written in a compact form

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0.$$

Recall, that $\partial_\mu = (\partial t, \vec{\nabla})$, while $x_\mu = (t, -\vec{x})$. Note, that the density trans-forms like the zero's component of four vector under a Lorentz boost, i.e $\rho = \gamma\rho_0$ where ρ_0 is the density in the rest-frame. Thus the four current of a fluid cell may be written as

$$j^\mu = \rho_0 u^\mu$$

where u^μ is the four velocity

$$u^\mu = (\gamma, \gamma\vec{\beta}) \tag{41}$$

with $\vec{\beta}$ the boost velocity and

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

Since a continuity equation holds for any conserved quantity, we can write them for the energy and each component of the momentum. If ϵ is the energy density, we obtain

$$\partial_t \epsilon + \vec{\nabla}(\vec{v}\epsilon) = \partial_t + \vec{\nabla}\vec{\mathcal{P}} = 0. \tag{42}$$

Here we have defined the momentum density $\vec{\mathcal{P}} = \epsilon\vec{v}$. Since each component of the three-momentum is conserved independently we obtain three more continuity equations which involve the momentum densities $\mathcal{P}^i$

$$\partial_t \mathcal{P}^i + \partial_j(v^j \mathcal{P}^i) = \partial_t \mathcal{P}^i + \partial_j \Pi^{ji} = 0. \tag{43}$$

Here, we have introduced the so called *momentum flow tensor*

$$\Pi^{ij} = v^i \mathcal{P}^j \tag{44}$$

which describes the momentum flow in a given direction. For example, Π^{xy} describes the flow of x-momentum in the y-direction. Obviously, off-diagonal elements of the stress tensor are a measure for the shear (or stress). Suppose

$\Pi^{xy} \neq 0$, then two layers stacked up in the y-direction moving with relative velocity in the x-direction will be subject to friction, because x-momentum flows from one layer to the other. This is nothing but shear as indicated in Fig.3[6]. For a system at rest in thermal equilibrium the stress tensor is diagonal

$$\Pi^{ij} = p\delta^{ij} \tag{45}$$

because any remaining shear forces would result in effectively rotating the system. The diagonal elements

$$\Pi^{ii} = v^i \mathcal{P}^i$$

are the momentum flow through the surface df^i, perpendicular to the flow direction. Consequently $\Pi^{ii} df^i = \frac{dp^i}{dt}$ is the force acting on this surface. This is what the pressure in a thermal system does so that the diagonal elements of the stress tensor are simply the pressure as indicated in Eq.45.

The continuity equations for the energy-current, 42, and momentum-current, 43, can be written in a compact form

$$\partial_\mu T^{\mu\nu} = 0 \tag{46}$$

where we have introduced the so-called *energy-momentum* tensor or *stress-energy* tensor $T^{\mu\nu}$ which has the following components

$$\begin{aligned} T^{00} &= \epsilon \\ T^{i0} = T^{0i} &= \mathcal{P}^i \ \ i = 1, 2, 3 \\ T^{ij} &= \Pi^{ij} \ \ i, j = 1, 2, 3. \end{aligned} \tag{47}$$

Equation 46 expresses the conservation of energy and momentum as a set of continuity equations as detailed above.

For a gas of free particles the energy-momentum tensor is given by [2]

$$T^{\mu\nu}(x) = \sum_i \frac{p_i^\mu p_i^\nu}{E_i} \delta^3(x_i - x)$$

where p_i^μ is the four-momentum of particle i, Here we also made explicit that the stress energy tensor is a field, i.e. it depends on the coordinates.

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[6]One has to be a little more careful here. There will be off-diagonal elements in Π^{ij} from motion of fuid cells. Stress, stress however, will also lead to off-diagonal elements in the rest frame of the cell.
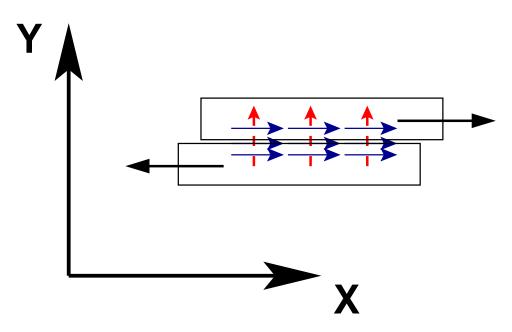
Figure 3: Effect of finite off-diagonal component Π^{xy}. The flow direction is indicated by the vertical red arrows and the momenta are given by the horizontal blue arrows.

Since hydrodynamics is nothing more than that plus the conservation of all other charges, the equations for relativistic ideal hydrodynamics are simply given by Eq.46 for a system where local thermal equilibrium is maintained. In addition the continuity equations for each conserved charge such as baryon number, strangeness etc need to be fulfilled.

4.1.2 Relativistic ideal hydrodynamics

Having all the necessary equations governing relativistic hydrodynamics, all we need is the appropriate form of the stress-energy tensor $T^{\mu\nu}$. To this end let us consider the stress-energy in the rest-frame of a fluid cell. In this case, the momentum densities vanish,

$$T^{i0} = T^{0i} = \mathcal{P}^i = 0 \ \ i = 1, 2, 3.$$

Furthermore, hydrodynamics assumes local thermal equilibrium. Following the previous discussion (Eq.45 and Eq.47) the spacial part of the tensor is

$$T^{ij} = \Pi^{ij} = p\delta^{ij} \ \ i, j = 1, 2, 3$$

so that

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

is diagonal and covered by the pressure and the energy density of the system at rest. Of course during the evolution the fluid cell starts moving so we need the stress-energy tensor of a moving fluid cell. Since we still assume, that the cell should be in local thermal equilibrium, the moving cell differs from one at rest only by the cells velocity i.e. by a Lorentz boost. Or in other word, a moving cell has a given four-velocity u_μ. Therefore, the general for of $T^{\mu\nu}$ for a cell moving with four-velocity u_μ is given by

$$T^{\mu\nu} = u^\mu u^\nu (\epsilon + p) - p \, g^{\mu\nu} \tag{48}$$

where

$$g = diag(1, -1 - 1 - 1)$$

is the metric tensor. Using the definition of the four-velocity Eq.41 we can easily verify that the general form of $T^{\mu\nu}$ reduces to that in the rest-frame

if cell does not move, i.e $u^\mu = (1, 0)$. The equations of ideal relativistic hydrodynamics are then

$$\partial_\mu T^{\mu\nu} = 0 \tag{49}$$
$$\partial j_i^\mu = 0 \tag{50}$$

i.e. the conservation of energy, momentum, and all charges i of the system.

Next let us write these equations more explicitely for the case of one conserved charge. In this case the continuity equation for the rest-frame density ρ_0 of the conserved charge is

$$\partial_\mu j^\mu = \partial_\mu(\rho_0 u^\mu) = 0 \tag{51}$$

Inserting the explicit form for $T^{\mu\nu}$, Eq.48 into Eq.49 we get

$$0 = \partial_\mu u^\mu u^\nu (\epsilon + p) - \partial^\nu p. \tag{52}$$

which are four independent equations. The most instructive way to look at these equations is to separate them into three spatial (vector) equation, and one scalar equation. The spatial equations will have the well know Euler equations of non-relativistic hydrodynamics as their limit. The scalar equation will be the continuity equation for the entropy flow. Lets first look at the spatial equations by setting $\nu = i = 1, 2, 3$

$$0 = \vec{u}\partial_\mu(u^\mu w) + (u^\mu w)\partial_\mu\vec{u} + \vec{\nabla}p \tag{53}$$

where we have set $w = \epsilon + p$, the enthalpy. For $\nu = 0$ we get

$$\begin{aligned} 0 &= u_0\partial_\mu(u^\mu w) + (u^\mu w)\partial_\mu u_0 - \partial_t p \\ (u^\mu w)\partial_\mu u_0 &= \partial_t p - u_0\partial_\mu(u^\mu w). \end{aligned} \tag{54}$$

Next we write $\vec{u} = u_o\vec{v}$ and consequently

$$\begin{aligned} (u^\mu w)\partial_\mu\vec{u} &= (u^\mu w)\vec{v}(\partial_\mu u_0) + (u^\mu w)u_o(\partial_\mu\vec{v}) \\ &= \vec{v}\left[\partial_t p - u_0\partial_\mu(u^\mu w)\right] + (u^\mu w)u_o(\partial_\mu\vec{v}) \end{aligned}$$

Inserting into Eq.53 the first term cancels and we get

$$\begin{aligned} 0 &= \vec{v}\partial_t p + (u^\mu w)u_o(\partial_\mu\vec{v}) + \vec{\nabla}p \\ &= \vec{v}\partial_t p + u_0^2 w\left[\partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] + \vec{\nabla}p. \end{aligned}$$

32

Reordering we get

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{wu_0^2}\left[\vec{\nabla}p + \vec{v}\partial_t p\right] = -\frac{1-v^2}{\epsilon+p}\left[\vec{\nabla}p + \vec{v}\partial_t p\right] \qquad (55)$$

where we have used that $u_0 = \gamma = \frac{1}{\sqrt{1-v^2}}$.

In the nonrelativistic limit, all terms $\sim v$ can be ignored. Furthermore, the energy density reduces to the mass density $\epsilon \rightarrow \rho_m$. In addition the pressure is in this case small compared to the mass density. We thus obtain

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho_m}\vec{\nabla}p \qquad (56)$$

which is the well known *Euler*-equation for non-relativistic ideal hydrodynamics.

So far we have three (vector) equations. The fourth, scalar one, we get by multiplying Eq.52 with the four-velocity u_ν. Using that

$$u_\mu u^\mu = 1$$
$$\partial_\nu\left(u_\mu u^\mu\right) = \frac{1}{2}u_\mu\partial_\nu u^\mu = 0$$

we get

$$0 = \partial_\mu(u^\mu w) - u_\nu \partial^\nu p.$$

Next we rewrite the first term of the previous equation identically as

$$u^\mu w = \rho_0 u^\mu \left(\frac{w}{\rho_0}\right) = j^\mu\left(\frac{w}{\rho_0}\right)$$

and use the continuity equation 51 to arrive at

$$0 = \rho_0 u^\mu \left[\partial_\mu\left(\frac{w}{\rho_0}\right) - \frac{1}{\rho_0}\partial_\mu p\right].$$

Next thermodynamics tells us (see appendix B) that the enthalpy density w is related to the entropy-density σ, particle density ρ_0 and pressure p via

$$d\left(\frac{w}{\rho_0}\right) = Td\left(\frac{\sigma}{\rho_0}\right) + \frac{1}{\rho_0}dp$$

Thus the above equation is simply a continuity equation for the entropy current

$$0 = \rho_0 u^\mu T \partial_\mu \left(\frac{\sigma}{\rho_0} \right)$$

Using the continuity equation 51 this can be cast into

$$\partial_\mu(\sigma u^\mu) = 0 \qquad (57)$$

which is simply a continuity equation for the entropy current

$$j_\sigma^\mu \equiv \sigma u^\mu \qquad (58)$$

The meaning of the continuity equation of the entropy current is nothing else that we are dealing with isentropic (adiabatic) dynamics in case of ideal hydro.

To summarize, the (five) equations governing ideal hydro with one conserved charge are

$$
\begin{aligned}
\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} &= -\frac{1 - v^2}{\epsilon + p} \left[\vec{\nabla} p + \vec{v} \partial_t p \right] \\
\partial_\mu(\sigma u^\mu) &= 0 \\
\partial_\mu(\rho_0 u^\mu) &= 0
\end{aligned}
\qquad (59)
$$

This set of differential equations are closed by the equation of state which for example gives the pressure as a function of entropy and density

$$p = p(\sigma, \rho_o)$$

In addition one needs initial conditions for the velocity, the entropy and the density. The result after hydro evolution is a velocity , entropy and density field or schematically

$$(\vec{v}(\vec{r}, t_0),\ \rho_0(\vec{r}, t_0),\ \sigma(\vec{r}, t_0)) \overset{Hydro}{\Longrightarrow} (\vec{v}(\vec{r}, t_{final}),\ \rho_0(\vec{r}, \vec{t}_{final}),\ \sigma(\vec{r}, t_{final}))$$

A   Four-vectors

When dealing with four-vectors we have to differentiate between contra-variant and co-variant four-vectors. A contra-variant four vector is given by

$$x^\mu = (t, x, y, z)$$

whereas the co-variant version is

$$x_\mu = (t, -x, -y, -z).$$

They are related by the metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = diag(1, -1, -1, -1)$$

$$x^\mu = g^{\mu\nu} x_\nu.$$

Furthermore

$$g^\nu_\mu = \delta^\nu_\mu = diag(1, 1, 1, 1)$$

and

$$x_\mu x^\mu = x^\mu x^\nu g_{\mu\nu}$$

is invariant under Lorentz transformations. When dealing with derivatives one has to be careful the contra-variant derivative is given by

$$\partial^\mu = (\frac{\partial}{\partial_t}, \frac{\partial}{\partial x_i}) = (\partial_t, -\vec{\nabla})$$

and the co-variant derivative is

$$\partial_\mu = (\partial_t, \vec{\nabla}).$$

Thus the spatial components enter with the opposite sign.

B   Some thermodynamics

The basic thermodynamic identity for a system with one conserved charge N is

$$E = TS - pV + \mu N \tag{60}$$

The first law of thermodynamics is expressed through the differential of the extensive quantities (E, S, V, N) via

$$dE = T\,dS - p\,dV + \mu\,dN.$$

This relation is also directly related to the micro-canonical ensemble, which also is controlled by the same macroscopic quantities. To go to the canonical ensemble, one introduces a Legendre transformation and introduces the free energy F

$$F = E - TS = -pV + \mu N$$

so that

$$dF = -S\,dT - p\,dV + \mu\,dN.$$

The free energy is related with the canonical partition function via

$$F = -T \ln Z.$$

The grand-canonical free energy often called the thermodynamic potential Ω is obtained by another Legendre transformation

$$\Omega = E - TS + \mu N = F + \mu N = -pV$$

and

$$d\Omega = -S\,dT - p\,dV - N\,d\mu.$$

The relation to the grand-canonical partition function is again

$$\Omega = -T \ln Z_G$$

from which follows, the relation between Z_G and the pressure p

$$pV = T \ln Z_G. \tag{61}$$

Another thermodynamic potential is the *enthalpy* H which controls a system where the volume may fluctuate but the pressure is kept constant, for example a balloon under water.

$$H = E + pV = TS + \mu N$$

and

$$dH = T\,dS + V\,dp + \mu\,dN.$$

36

In the context of hydrodynamics, one deals with the enthalpy density

$$w = \frac{H}{V} = \epsilon + p = T\sigma + \mu\rho$$

where $\epsilon = \frac{E}{V}$ is the energy density, $\sigma = \frac{S}{V}$ the entropy density and $\rho = \frac{N}{V}$ the particle density. For the differential we have

$$dw = T\,d\sigma + dp + \mu\,d\rho$$

In the discussion of hydrodynamics we need the differential $d(\frac{w}{\rho})$ which we will work out here

$$
\begin{aligned}
d(\frac{w}{\rho}) &= \frac{1}{\rho}\left(dw - w\frac{d\rho}{\rho}\right) \\
&= \frac{1}{\rho}\left(T\,d\sigma + dp + \mu\,d\rho - T\frac{\sigma}{\rho}d\rho - \mu\,d\rho\right) \\
&= T(\frac{d\sigma}{\rho} - \frac{\sigma}{\rho^2}d\rho) + \frac{1}{\rho}dp \\
&= T\,d\left(\frac{\sigma}{\rho}\right) + \frac{1}{\rho}dp
\end{aligned}
$$

References

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