Mechanical Oscillations

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1 Reading:

Shamos, Great Experiments in Physics, pp. 42-58

2 Harmonic Motion

2.1 Free Oscillator

Many different types of harmonic motion exist. Of these, the most basic is called "simple harmonic motion", which is purely sinusoidal and therefore easily analyzed mathematically. Simple harmonic motion exits in an oscillating system when the restoring force of the system is directly proportional to the displacement. Thus Newton's Second Law becomes

$$m\frac{d^2x}{dt^2} = -kx \tag{1}$$

where m is the mass, d^2x/dt is the acceleration, k is the force constant of the restoring force, x is the amount of displacement of the system, and t is the time. For an angular pendulum, k is the torsional[1] spring constant and m is the moment of inertia. The solution of this equation is of the form $x = A\sin\omega_0 t$, where ω_0 is the angular frequency in radians per second. Subsituting this into Eq. 1, we find that

$$m\omega_0^2 = k \tag{2}$$

which gives us the natural frequency of the oscillator

$$\omega_0 = \sqrt{k/m} \tag{3}$$

We can convert ω_0 into the frequency in hertz by noting that there are 2π radians per cycle or $\omega = 2\pi f$

$$f_0 = (1/2\pi)\sqrt{k/m} \tag{4}$$

The period is just the inverse of the frequency.

If a simple harmonic oscillator were perfectly isolated, it would continue to oscillate forever, because there would be no resistance to motion. In real systems, however, there is always some amount of resistance or friction which leads to damping of the oscillator. Therefore the model of simple harmonic oscillators is not adequate, and we need to add another term to the equation, namely a damping force which is proportional and opposite to the velocity of the oscillator, to form a more accurate model. The damped harmonic oscillator is described by:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{kx}{m} = 0 \tag{5}$$

[1] torsion comes from the Latin verb **torquēre** , to twist, whence the word torque

where b is the damping constant for the system. Such a differential equation has a solution of the form $x = e^{\alpha t}$. This substitution produces the characteristic equation for Eq. 5.

$$\alpha^2 + (b/m)\,\alpha + k/m = 0 \tag{6}$$

The solution to this quadratic equation is

$$\alpha = \left[-b/m \pm \sqrt{b^2/m^2 - 4k/m}\right]/2 \tag{7}$$

By examining the discriminant, $b^2/m^2 - 4k/m$, we find three different types of damping. If $b^2 = 4mk$, the equation has real and equal roots, and the system is said to be critically damped. In this case, the system approaches the equilibrium position, x=0, as rapidly as possible. If $b^2 > 4mk$, the equation has real and unequal roots, and the system is said to be overdamped. The system also approaches equilibrium without oscillation, but not so rapidly. If $b^2 < 4mk$, the equation has complex roots, and the system is said to be underdamped. The system will oscillate with decreasing amplitude. The solution for this case is

$$x\,(t) = e^{-\lambda t}\,(A\cos\omega' t + B\sin\omega' t) \tag{8}$$

The frequency of the damped oscillation is

$$f' = 1/(2\pi)\sqrt{(k/m - b^2/4m^2)} \tag{9}$$

According to this equation, an underdamped oscillator has a lower frequency than an undamped one, i.e., there is a frequency shift. For modest damping such that the amplitude of oscillation is still appreciable after a few oscillations, the shift is small but measurable. In this laboratory session we will be working with underdamped systems which begin oscillating at the maximum displacement, so that B = 0. It is straightforward to substitute this solution into Eq. 5 to verify that this is a solution for

$$\begin{aligned} \omega_0 &= k/m \\ \lambda &= \frac{b}{2m} \\ \omega' &= \sqrt{(\omega_o^2 - \lambda^2)} \end{aligned} \tag{10}$$

2.2 Driven oscillator

When a vibrating string is subjected to a periodic external force, it is said to be in a state of forced harmonic oscillation, which can be described by the differential equation.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F\cos\omega t. \tag{11}$$

where $F\cos\omega t$ is the driving force of the system. Note that we now have three omegas: $\omega_\circ, \omega', and \omega$. If the frequency of the periodic external force, ω, is equal to or very near the frequency of the damped oscillator, ω', then the system is in resonance, and the amplitude of the oscillations is large. The point at which this occurs is sometimes called a resonant peak because of the shape of the graph of amplitude versus frequency.

The general solution of Eq. 11 is the sum of two pieces: a) the solution to Eq. 5, which is the transient part, being damped out for $t \gg \lambda = 2m/b$, and b) the steady state solution

$$x = \frac{(F/m)\cos(\omega t + \phi)}{\sqrt{\left[(\omega^2 - \omega'^2)^2 + 4\lambda^2\omega_0^2\right]}} \tag{12}$$

The amplitude of oscillation is proportional to

$$\frac{1}{\sqrt{\left[(\omega^2 - \omega'^2)^2 + 4\lambda^2\omega_0^2\right]}},$$

which is maximum for an imposed frequency $\omega = \omega'$, and half as great for an angular frequency ω_h

$$\left(\omega_h^2 - \omega'^2\right)^2 = 12\lambda^2\omega_0^2$$

or

$$\omega_h^2 = \omega'^2 \pm 2\sqrt{3}\lambda\omega_0 \tag{13}$$

These equations can be used to describe, or at least approximate, nearly all naturally vibrating phenomena, and examination of machines which illustrate these properties, such as the torsional pendulum, will illustrate these concepts.

3 Apparatus

	Manufacturer
1 torsional pendulum apparatus	(Leybold-Heraeus)
2 photogates with timers	(Pasco Scientific)
1 15 V, 2 A power supply for damping coil	(GW Mod GPS 1850)
1 24 V, 0.8 A power supply to drive motor	(ElectroProducts)
test leads with banana plugs	
1 digital voltmeter	
1 digital ammeter	

4 Methods

The timing of the pendulum has a long history of tedium. The experiment requires measurements of the period of oscillation, which technology has fortunately helped us to do without the use of stopwatches and the counting of pendulum swings. We use a photogate, a device which uses a beam of light to control an accurate electronic timer. The settings on the base of the timer determine how the beam controls the timer. If the setting is "pend", then the timer will start when the light beam is broken the first time and will stop when the light beam is broken the third time, corresponding to one full period of a pendulum swing. When the setting is "pulse", the timer starts on one break and ends on the next, corresponding to one half of a period of a pendulum, if the gate is accurately centered. When it is set to "gate", the timer runs as long as the beam is broken, ending as soon as the obstruction is removed. We use "pend" mode so that the gate need not be at the midpoint of the swing.

5 Procedure for the Torsional Pendulum

Objective:
(a) measure the frequency and damping parameters for several intensities of braking
(b) plot the resonance curve for driven oscillator for several damping currents.

1. General setup procedures. A variable speed motor and connecting rod provide a nearly sinusoidal driving motion for a wheel, and a coil spring provides the restoring torque. Damping is provided by an electromagnet; a conductor moving through a magnetic field experiences a braking force. Connect the DC output of the ElectroProducts power supply to the two upper plugs on the driving motor control block on the right side of the apparatus. Use a digital voltmeter to measure the applied voltage and adjust the power supply to about 25 V. Applying higher voltages will burn out the resistor network which controls the speed of the motor. Adjustments to motor speed should be made with the coarse (gross) and fine (fein) controls on the box. Reconnect the voltmeter to the lower connections on the box. This connection measures the voltage applied to the motor and is more valuable for returning the motor speed to a value you had previously used and recorded speed and voltage in your notes.

The motion of the pendulum is damped by the electromagnetic damping coil located beneath the pendulum, with its connections in the back. Set the voltage knob on the GW supply counterclockwise (zero). Connect the DC output of the GW power supply in series with a digital ammeter and the damping coil.

Note: NEVER connect an ammeter in PARALLEL with a voltage source. Ammeters have a very low internal resistance and hence AMMETERS MUST BE CONNECTED IN SERIES WITH THE VOLTAGE SOURCE AND THE LOAD. To do otherwise could destroy the ammeter or blow its fuse, if it has any. This is why we ask you to have the instructor check the circuits you build before you turn them on.

Position the one photogate timer so that the extension of the horizontal connecting rod between the motor and the pendulum completely crosses the timer's beam twice when the driving wheel makes one revolution, that is, not off the end of the rod. Mount the other timer on the blocks of wood over the coil connections so that the piece taped to the wheel crosses the timer beam. One of these timers will measure the period of the pendulum and the other measures ω', the period of the driving force. The two periods will be identical once equilibrium has been reached, and the times will agree; in practice, you may find that the two timers do not agree perfectly. You will obtain erratic readings from the timers if the photogates are blocked at the instant when you push the start button - be careful to identify these readings and repeat the measurements whenever necessary.

2. Free Oscillations

   (a) Either by just turning the driving wheel by hand or by turning the ElectroProducts supply on and off, adjust the stopping position of the motor so that the white arrow on the pendulum points to zero with the motor off. This adjustment should center the equilibrium position of the wheel. Turn the damping current off for the first measurements.

   (b) Investigate the transient behavior of the oscillator with a step external force by using your fingers to move the pendulum ten units to one side prior to releasing it. Note that a slightly different period is obtained if amplitudes greater than ten are used. This a probable indication of departures from Hooke's Law for larger amplitudes, and you should not use data with larger amplitudes.

   (c) Measure the period of the oscillation with the photogate. Count the number of oscillations required for the amplitude to decrease to a convenient fraction of the initial value, such as 1/2 or 1/10. Use these data to determine $t_{1/2}$ or $t_{1/10}$ and hence λ. If only one or two oscillations are required to get to one half of the amplitude, your value of $t_{1/2}$ will be inaccurate and you must use $t_{1/10}$. Compute λ and ω' from these data. Repeat for damping currents of 0.10 A, 0.25 A, 0.45 A and 0.6 A.

   (d) Tabulate your values for λ and make a plot of λ versus the damping current. For each value of λ, what is the expected frequency shift, $\omega_0 - \omega'$, and how does it compare with your measurements? (How can you answer this question when you aren't given $\omega_\circ$?)

   (e) The magnet coil will overheat if currents greater than 1 A are used for more than a few minutes, but higher currents are needed to show the overdambed behavior. Observe the motion of the pendulum when currents of 1 and 1.5 A are used, but do not leave these high currents on for more than a minute or so. Describe the behavior of the pendulum.

3. Forced oscillations.

   (a) Set the damping current to the second value of 0.25 A and turn on the driver. Wait long enough for the two periods to be equal, so that the transient part of the solution has died out. Plot the amplitude of the oscillation versus frequency for driving frequencies between 0.45 Hz and 0.65 Hz (t between 2.222 s and 1.538 s). You will need steps of 0.005 Hz near the peak, while coarser steps are adequate at the edges. From the peak of your plot, find ω'. Note whether the driver and oscillation are in or out of phase for frequencies well above and well below resonance.

(b) Calculate the expected half-height points from your data in the preceding section and equation 13 or from the corresponding equation for $t_{1/10}$, which you derive. Plot these points as vertical lines on your graph of amplitude versus frequency.

(c) If time permits, repeat steps 3a and 3b for the third value of damping current, 0.45 A. This curve will be wider, so the range of frequency should be somewhat greater and the steps a bit larger.

6 Results

With what accuracy did you determine ω, ω', and λ? In the torsional pendulum, what determines the resonant frequency? What causes the damping? What is the distinguishing characteristic of the graphs when the driving frequency approaches the natural frequency? What happens when damping is applied? How does the width of the resonance curve depend on the amount of damping? Why does the frequency shift in a damped system? For what current is the system overdamped, underdamped, and critically damped? What are the sources of error in this experiment, and how could you build a better apparatus to minimize these?

7 References

1. Douglas C. Giancoli, General Physics (Englewood Cliffs: Prentice Hall, Inc. 1984)
2. B. Saraf, Physics through Experiment (Delhi: University Leadership Project in Physics, University of Rajasthan, Jaipur, 1979